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Eigenvalue Sensitivities of a Linear Structure Carrying Lumped Attachments

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The eigenvalue sensitivities of a combined system consisting of an arbitrarily supported linear structure carrying various lumped attachments are thoroughly analyzed in this paper. An efficient approach is proposed to determine the eigenvalue sensitivities with respect to the physical parameters of the lumped elements and their attachment locations using the implicit function theorem. Once found, these eigenvalue sensitivities can also be exploited to compute the perturbed eigenvalues when the combined system is subjected to slight modifications. The proposed method to determine the eigenvalue sensitivities is easy to formulate, systematic to apply, and simple to code. Numerical experiments validated the proposed scheme and showed that it leads to sensitivities that agree well with the exact results.

Nomenclature

a	=	length of a plate along the x direction
$[B]$	=	$p \times p$ characteristic matrix
b	=	length of a plate along the y direction
b_{ij}	=	(i, j)th element of $[B]$
c	=	viscous damping coefficient
D	=	flexural rigidity of a plate
E	=	Young's modulus
g	=	user-defined function that depends on the characteristic equation of the combined system
h	=	thickness of a plate
I	=	area moment of inertia of the cross section of a beam
$[I]$	=	identity matrix
j	=	imaginary unit
K_i	=	ith generalized stiffness of a host linear structure
K_{pq}	=	generalized stiffnesses of a host plate
$[K^d]$	=	$N \times N$ diagonal stiffness matrix whose elements are K_i
k	=	translational spring constant
k_t	=	torsional spring constant
L	=	length of a beam
M_i	=	ith generalized mass of a host linear structure
M_{pq}	=	generalized masses of a host plate
$[M^d]$	=	$N \times N$ diagonal mass matrix whose elements are M_i
m	=	lumped mass parameter
N	=	number of generalized coordinates
p	=	number of lumped attachments
q	=	number of attachment locations
s	=	number of parameters that characterizes a lumped attachment
t	=	time variable
$\mathbf{u}(x_a)$	=	$[u_1(x_a)\ u_2(x_a)\ \cdots\ u_N(x_a)]^T$
$\mathbf{u}_i$	=	$\mathbf{u}(x_i)$
$u_i(x)$	=	a function that depends on the ith eigenfunction of a linear structure evaluated at x
$w(x, t)$	=	lateral deflection of a linear structure evaluated at x and t
x	=	spatial coordinate along a linear structure
x_a	=	attachment location in the x direction
x_i	=	ith attachment location
y_a	=	attachment location in the y direction
α_i	=	ith physical parameter of a lumped attachment
$\mathbf{\Delta}$	=	vector of the perturbations of the system parameters and attachment locations
$\Delta\alpha_r$	=	change in the rth lumped system parameter from its nominal value
Δx_s	=	change in the sth attachment location from its nominal value
δ_i^j	=	Kronecker delta
$\eta_i(t)$	=	ith generalized coordinate evaluated at t
$\mathbf{\Lambda}$	=	column vector of the perturbed eigenvalues
$[\Lambda]$	=	diagonal matrix consisting of the exact eigenvalues of a linear structure
$\mathbf{\Lambda_0}$	=	column vector of the unperturbed eigenvalues
λ	=	eigenvalue of the combined system
λ_I	=	imaginary part of λ
λ_i	=	ith eigenvalue of the combined system or the ith eigenvalue of the perturbed system
λ_{i0}	=	ith eigenvalue of the unperturbed system
λ_R	=	real part of λ
μ	=	Poisson's ratio
ρ	=	mass per unit length of a beam or mass per unit area of a plate
σ	=	a parameter that describes the lumped attachment
σ_i	=	parameter that describes the ith lumped attachment
$\phi_i(x)$	=	ith eigenfunction of a host linear structure evaluated at x
$\phi_i'(x)$	=	derivative of $\phi_i(x)$ with respect to x
ω	=	natural frequency of a combined system
ω_{hi}	=	ith natural frequency of a host linear structure
$[\nabla\lambda]$	=	matrix of the eigenvalue gradients

I. Introduction

THE dynamic analysis of mechanical systems often leads to the solution of an eigenvalue problem. Since most physical structures are complicated, in order to meet design specifications, modifications are often necessary and will generally alter the eigenvalues of the system. Thus, efficient methods of calculating the partial derivatives of the eigenvalues with respect to some physical parameters of the system are frequently desired. The calculation of these partial derivatives is referred to as sensitivity analysis.

The sensitivities of the eigenvalues with respect to arbitrary parameters of structural systems have received considerable interest over the years. Significant developments and contributions are due to

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many different researchers, and hence only a few selected references are cited here [1–14]. In this paper, the approximate eigenvalue sensitivities of a combined vibratory system are thoroughly investigated, where the combined system consists of a linear structure carrying various undamped or damped attachments. Wang [15] derived the approximate sensitivities of the eigenfrequencies of linear structures with respect to changes in the position of attached masses, restraining springs and spring-mass oscillators by using the normal mode method. Gürgöze et al. [16] studied the sensitivities of the eigenvalues with respect to small changes in the location of the in-span support for a cantilevered beam with a tip mass and an in-span support. Gürgöze [17] also analyzed the eigenvalue sensitivities of a viscously damped cantilever carrying a tip mass. In [16,17], the authors first derived the characteristic equation of the system via a boundary value formulation and then partially differentiated the frequency equation to obtain the exact eigenvalue sensitivities with respect to the physical parameter of interest. While conceptually simple, taking the partial derivatives directly with respect to a chosen physical parameter can be nontrivial. For a complicated system consisting of a linear structure carrying multiple lumped attachments, establishing the exact frequency equation and computing the exact eigenvalue sensitivities can often be difficult, if not impossible.

In this paper, the approximate eigenvalue sensitivities of an arbitrarily supported linear structure carrying various lumped attachments are analyzed. Using the approach described in the next section, the approximate frequency equation can be obtained in closed form, and the results can be easily summarized in lookup tables. Once the approximate frequency equation is established, a simple scheme is proposed to determine the approximate eigenvalue sensitivities, thus allowing design engineers to readily compute the sensitivities of the eigenvalues with respect to any physical parameters. Numerical case studies are presented to illustrate the validity of the proposed method and, whenever possible, the approximate eigenvalue sensitivities are compared with those obtained exactly.

II. Theory

In theory, the eigenvalue sensitivities of a combined system consisting of a linear structure carrying a single or multiple attachments can be solved exactly. First, the exact frequency equation is obtained via a boundary value formulation. Then the eigenvalue sensitivities are determined by means of direct differentiation of the exact frequency equation. In application, however, formulating the exact frequency equation can be tedious, and partially differentiating the exact frequency equation is often nontrivial, especially if the linear structure is carrying multiple attachments. In this section, an approximate frequency equation will be derived first, and then the approximate eigenvalue sensitivities will be established via simple partial differentiations by exploiting the implicit function theorem.

A. Single Attachment

Consider an arbitrarily supported linear structure carrying a single lumped attachment, which may include a lumped mass or rotary inertia, a grounded translation or rotational viscous damper, a grounded translation or rotational spring, an undamped or damped oscillator with or without a rigid-body degree of freedom, or the Maxwell model for a viscoelastic solid [18]. Using the assumed-modes method [19], the lateral deflection of the linear structure can be expressed in the form of a finite series as follows:

$$w(x,t)=\sum_{i=1}^{N}\phi_i(x)\eta_i(t) \tag{1}$$

Formulating the total kinetic and potential energies (and Rayleigh's dissipation function if the linear structure carries any viscously damped attachment) of the combined system and then applying Lagrange's equations, one finds that the eigenvalues of the system are given by the solution of the following characteristic equation (see [20] for detailed derivations):

$$\det(\lambda^2[M^d]+[K^d]+\sigma\mathbf{u}(x_a)\mathbf{u}^T(x_a)) = \left(\prod_{i=1}^{N}(\lambda^2 M_i+K_i)\right)\left(1+\sigma\sum_{i=1}^{N}\frac{u_i^2(x_a)}{\lambda^2 M_i+K_i}\right)=0 \tag{2}$$

Both $\mathbf{u}(x_a)$ and σ depend on the element type, and Table 1 summarizes the expressions for $\mathbf{u}(x_a)$ and σ for the various lumped attachments [20]. If the eigenfunctions of the host structure are normalized with respect to its mass per unit length, then

$$M_i=1, \qquad K_i=\omega_{hi}^2 \tag{3}$$

In this case, Eq. (2) becomes

$$\left(\prod_{i=1}^{N}(\lambda^2+\omega_{hi}^2)\right)\left(1+\sigma\sum_{i=1}^{N}\frac{u_i^2(x_a)}{\lambda^2+\omega_{hi}^2}\right)=0 \tag{4}$$

For a translational (or rotational) lumped attachment, when x_a does not coincide with a point of zero translational (or angular) displacement of any eigenfunction of the linear structure, all the eigenvalues of the combined system will be distinct from those of the host linear structure. For such a combined system, $\lambda^2+\omega_{h_i}^2\neq 0$, and Eq. (4) reduces to

$$1+\sigma\sum_{i=1}^{N}\frac{u_i^2(x_a)}{\lambda^2+\omega_{hi}^2}=0 \tag{5}$$

Equation (5) is known as the frequency equation of the system. In application, it is nearly impossible to have the attachment location for a translational or rotational element to coincide exactly with a point of zero displacement or a point of zero slope of the normal modes. Thus, it will be assumed that $\lambda^2+\omega_{hi}^2\neq 0$ in the subsequent analysis. For a given lumped element and attachment location, the eigenvalues of the system correspond to the zeros of Eq. (5), which can be determined either graphically or numerically using any root solver routines, such as `fsolve` in MATLAB or `zeroin` in EISPACK. Finally, for an undamped system, $\lambda=j\omega$; for a damped system, $\lambda=\lambda_R+j\lambda_I$.

Eigenvalue sensitivities with respect to the physical parameters of the lumped element and its attachment location can be determined by the direct differentiation of Eq. (5). However, depending on the lumped element type, computing the eigenvalue sensitivities directly may be laborious, because the eigenvalues depend on all the physical parameters of the attachment. Here, an alternative method based on the implicit function theorem (see [21] or Appendix A) will be used to efficiently calculate the eigenvalue sensitivities. To this end, a function g is defined as follows:

$$g=1+\sigma\sum_{i=1}^{N}\frac{u_i^2(x_a)}{\lambda^2+\omega_{hi}^2} \tag{6}$$

Mathematically, σ can be expressed as

$$\sigma=\sigma(\alpha_1,\alpha_2,\ldots,\alpha_s) \tag{7}$$

Note that by setting g equal to zero, one obtains the frequency equation of the system. Once the function g has been defined, the implicit function theorem can be applied to determine the eigenvalue sensitivities, which are given by

$$\frac{\partial\lambda}{\partial\alpha_i}=-\frac{\partial g}{\partial\alpha_i}\bigg/\frac{\partial g}{\partial\lambda} \tag{8}$$

$$\frac{\partial\lambda}{\partial x_a}=-\frac{\partial g}{\partial x_a}\bigg/\frac{\partial g}{\partial\lambda} \tag{9}$$

where $\partial g/\partial\alpha_i$, $\partial g/\partial x_a$, and $\partial g/\partial\lambda$ can be easily obtained by taking the partial derivatives of g. It should be noted that the eigenvalue sensitivities are defined if and only if $\partial g/\partial\lambda \neq 0$. For a single attachment, this requirement is met provided the combined system does not execute a rigid-body mode and if the attachment location for a translational (or rotational) attachment does not coincide with any point of zero displacement (or slope) of the eigenfunctions of the host linear structure.

Determining the eigenvalue sensitivities using the implicit function theorem is simple, because in taking the partial derivative of g with respect to any desired physical parameter, one assumes all of the remaining parameters to be constant. For example, consider a linear structure carrying a lumped mass. From Table 1, $\sigma = m\lambda^2$ and the desired g function is given by

$$g = 1 + m\lambda^2 \sum_{i=1}^{N} \frac{\phi_i^2(x_a)}{\lambda^2 + \omega_{hi}^2} \tag{10}$$

For a lumped mass, $\alpha = m$, and taking the partial derivatives of g with respect to λ, α and x_a results in the first row of Table 2. The other rows of Table 2 are similarly obtained. If σ is a rational function (see Table 1), it is often convenient to first multiply the characteristic equation of Eq. (5) by the denominator polynomial of σ, and define the resulting function as g. For example, for a linear structure carrying a damped translational oscillator with a rigid degree of freedom, its frequency equation is given by

Table 1 Expression for σ and u for any lumped attachment at x_a[a]

Lumped attachment	σ	**u**
Lumped mass	$m\lambda^2$	ϕ
Rotary inertia	$J\lambda^2$	ϕ'
Grounded translational spring	k	ϕ
Grounded rotational spring	k_r	ϕ'
Grounded translational viscous damper	$c\lambda$	ϕ
Grounded rotational viscous damper	$c_r\lambda$	ϕ'
Maxwell model for a viscoelastic solid	$ck\lambda/(c\lambda + k)$	ϕ
Damped oscillator with no rigid DOF	$k + c\lambda + m\lambda^2$	ϕ
Damped oscillator with a rigid DOF	$(k + c\lambda)m\lambda^2/$ $(k + c\lambda + m\lambda^2)$	ϕ
In-span simple support	$k \to \infty$	ϕ

[a]DOF stands for degree of freedom, and the prime denotes a derivative with respect to x.

$$1 + \frac{(k + c\lambda)m\lambda^2}{k + c\lambda + m\lambda^2} \sum_{i=1}^{N} \frac{\phi_i^2(x_a)}{\lambda^2 + \omega_{hi}^2} = 0 \tag{11}$$

and the desired function g is defined as

$$g = k + c\lambda + m\lambda^2 + (k + c\lambda)m\lambda^2 \sum_{i=1}^{N} \frac{\phi_i^2(x_a)}{\lambda^2 + \omega_{hi}^2} \tag{12}$$

whose partial derivatives with the system parameters and attachment location can be easily computed. Table 2 and Appendix B summarize the partial derivatives of g with respect to the relevant physical parameters associated for the set of lumped attachments shown in Table 1, assuming that the eigenfunctions have been properly normalized so that Eq. (3) is satisfied. Because these partial derivatives are tabulated, nonexperts can immediately obtain the eigenvalue sensitivities without having to perform any analysis.

B. Multiple Attachments

Table 1 can be extended to an arbitrarily supported linear structure carrying p lumped attachments at distinct locations x_i $(i = 1, \ldots, p)$, where the corresponding σ_i and $\mathbf{u}_i$ are obtained directly from Table 1. For this general case, the eigenvalues are given by the solution of the following characteristic determinant [20]:

$$\det\left(\lambda^2[M^d] + [K^d] + \sum_{i=1}^{p} \sigma_i \mathbf{u}_i \mathbf{u}_i^T\right) = 0 \tag{13}$$

Assuming all the eigenvalues of the combined system are distinct from those of the host structure, Eq. (13) reduces to [20]:

$$\det[B] = 0 \tag{14}$$

where

$$b_{ij} = \sum_{r=1}^{N} \frac{u_r(x_i)u_r(x_j)}{\lambda^2 M_r + K_r} + \frac{1}{\sigma_i}\delta_i^j, \qquad i, j = 1, 2, \ldots, p \tag{15}$$

The eigenvalues of the system with multiple attachments correspond to the roots of Eq. (14).

For the general case of multiple attachments, the function g is defined as

$$g = \det[B] \tag{16}$$

Table 2 Partial derivatives of g for various lumped attachments[a]

α	$\partial g/\partial\lambda$	$\partial g/\partial\alpha$	$\partial g/\partial x_a$
m	$2m\lambda \sum_{i=1}^{N} \frac{\omega_{hi}^2\phi_i^2(x_a)}{(\lambda^2+\omega_{hi}^2)^2}$	$\lambda^2 \sum_{i=1}^{N} \frac{\phi_i^2(x_a)}{\lambda^2+\omega_{hi}^2}$	$2m\lambda^2 \sum_{i=1}^{N} \frac{\phi_i(x_a)\phi_i'(x_a)}{\lambda^2+\omega_{hi}^2}$
J	$2J\lambda \sum_{i=1}^{N} \frac{\omega_{hi}^2\phi_i'^2(x_a)}{(\lambda^2+\omega_{hi}^2)^2}$	$\lambda^2 \sum_{i=1}^{N} \frac{\phi_i'^2(x_a)}{\lambda^2+\omega_{hi}^2}$	$2J\lambda^2 \sum_{i=1}^{N} \frac{\phi_i'(x_a)\phi_i''(x_a)}{\lambda^2+\omega_{hi}^2}$
c	$c \sum_{i=1}^{N} \frac{(\omega_{hi}^2-\lambda^2)\phi_i^2(x_a)}{(\lambda^2+\omega_{hi}^2)^2}$	$\lambda \sum_{i=1}^{N} \frac{\phi_i^2(x_a)}{\lambda^2+\omega_{hi}^2}$	$2c\lambda \sum_{i=1}^{N} \frac{\phi_i(x_a)\phi_i'(x_a)}{\lambda^2+\omega_{hi}^2}$
c_r	$c_r \sum_{i=1}^{N} \frac{(\omega_{hi}^2-\lambda^2)\phi_i'^2(x_a)}{(\lambda^2+\omega_{hi}^2)^2}$	$\lambda \sum_{i=1}^{N} \frac{\phi_i'^2(x_a)}{\lambda^2+\omega_{hi}^2}$	$2c_r\lambda \sum_{i=1}^{N} \frac{\phi_i'(x_a)\phi_i''(x_a)}{\lambda^2+\omega_{hi}^2}$
k	$-2k\lambda \sum_{i=1}^{N} \frac{\phi_i^2(x_a)}{(\lambda^2+\omega_{hi}^2)^2}$	$\sum_{i=1}^{N} \frac{\phi_i^2(x_a)}{\lambda^2+\omega_{hi}^2}$	$2k \sum_{i=1}^{N} \frac{\phi_i(x_a)\phi_i'(x_a)}{\lambda^2+\omega_{hi}^2}$
k_r	$-2k_r\lambda \sum_{i=1}^{N} \frac{\phi_i'^2(x_a)}{(\lambda^2+\omega_{hi}^2)^2}$	$\sum_{i=1}^{N} \frac{\phi_i'^2(x_a)}{\lambda^2+\omega_{hi}^2}$	$2k_r \sum_{i=1}^{N} \frac{\phi_i'(x_a)\phi_i''(x_a)}{\lambda^2+\omega_{hi}^2}$
$k \to \infty$	$-2\lambda \sum_{i=1}^{N} \frac{\phi_i^2(x_a)}{(\lambda^2+\omega_{hi}^2)^2}$	NA	$2 \sum_{i=1}^{N} \frac{\phi_i(x_a)\phi_i'(x_a)}{\lambda^2+\omega_{hi}^2}$

[a] The prime denotes a derivative with respect to x.

Consider the case where $p = 2$ (a linear structure carrying two attachments). For simplicity, assume that the eigenfunctions of the bare structure are properly normalized such that Eq. (3) is satisfied. Expanding Eq. (16), one obtains

$$g = 1 + \sigma_1\sigma_2\left(\sum_{i=1}^{N}\frac{u_i^2(x_1)}{\lambda^2+\omega_{hi}^2}\right)\left(\sum_{i=1}^{N}\frac{u_i^2(x_2)}{\lambda^2+\omega_{hi}^2}\right) + \sigma_1\sum_{i=1}^{N}\frac{u_i^2(x_1)}{\lambda^2+\omega_{hi}^2} + \sigma_2\sum_{i=1}^{N}\frac{u_i^2(x_2)}{\lambda^2+\omega_{hi}^2} - \sigma_1\sigma_2\left(\sum_{i=1}^{N}\frac{u_i(x_1)u_i(x_2)}{\lambda^2+\omega_{hi}^2}\right)^2 \tag{17}$$

where σ_i and u_i depend on the type of lumped attachments (see Table 1). To determine the sensitivities of the eigenvalues, the implicit function theorem is again used. For example, the eigenvalue sensitivities with respect to the two attachment locations are given by

$$\frac{\partial\lambda}{\partial x_1} = -\frac{\partial g}{\partial x_1}\bigg/\frac{\partial g}{\partial\lambda}, \qquad \frac{\partial\lambda}{\partial x_2} = -\frac{\partial g}{\partial x_2}\bigg/\frac{\partial g}{\partial\lambda} \tag{18}$$

As before, the eigenvalue sensitivities are only defined for $\partial g/\partial\lambda \neq 0$, i.e., when the combined system does not execute a rigid-body mode and when all the attachment locations for the translational (or rotational) elements are distinct from points of zero displacement (or slope) of any eigenfunction of the host structure.

C. Perturbed Eigenvalues

Eigenvalue sensitivities can also be used to determine the perturbed eigenvalues of a combined system that is subjected to slight modifications. A frequently encountered scenario in structural dynamics is determining the perturbed eigenvalues of a system after certain modifications are introduced. Clearly, if these modifications are substantial, then a new analysis and computational cycle are necessary in order to compute these new or perturbed eigenvalues. However, if the changes made are small, then the eigenvalue sensitivities previously defined can be exploited to determine accurate approximations to the perturbed eigenvalues of the modified system without performing a new analysis.

Consider a combined system consisting of a linear structure carrying multiple attachments. Its ith eigenvalue depends on the physical parameters and the attachment locations as follows:

$$\lambda_i = \lambda_i(\alpha_1, \ldots, \alpha_p, x_1, \ldots, x_q) \tag{19}$$

To the first-order approximation, Taylor series expansion of Eq. (19) yields

$$\lambda_i \approx \lambda_{i0} + \left(\frac{\partial\lambda_i}{\partial\alpha_1}\right)_0\Delta\alpha_1 + \cdots + \left(\frac{\partial\lambda_i}{\partial\alpha_p}\right)_0\Delta\alpha_p + \left(\frac{\partial\lambda_i}{\partial x_1}\right)_0\Delta x_1 + \cdots + \left(\frac{\partial\lambda_i}{\partial x_q}\right)_0\Delta x_q \tag{20}$$

In Eq. (20), all the partial derivatives are evaluated at λ_{i0} and at the nominal system parameters and attachment locations. For a given combined system, assume that N modes are sufficient to calculate the first N' eigenvalues accurately, where $N' < N$. Then for slight modifications, $\Delta\alpha_r$ and Δx_s, Eq. (20) can be used to find accurate approximations to the first N' perturbed eigenvalues. Equation (20), for $i = 1, \ldots, N'$, can be expressed more compactly as

$$\mathbf{\Lambda} \approx \mathbf{\Lambda_0} + [\nabla\lambda]\mathbf{\Delta} \tag{21}$$

where $\mathbf{\Lambda}$ and $\mathbf{\Lambda_0}$ are both of length N'; $[\nabla\lambda]$ is a $N' \times (p+q)$ matrix whose ith row consists of the gradients $(\partial\lambda_i/\partial\alpha_r)_0$ (for $r = 1, \ldots, p$) and $(\partial\lambda_i/\partial x_s)_0$ (for $s = 1, \ldots, q$), evaluated at the nominal or unperturbed values. $\mathbf{\Delta}$, of length $p + q$, is a column vector containing the perturbations of the system parameters and attachment locations. By calculating all the eigenvalue sensitivities and evaluating them at their unperturbed values, one can use Eq. (21) to find accurate approximations of the first N' perturbed eigenvalues without performing a costly and completely new cycle of calculations.

III. Results

The main contribution of the present paper is to develop a systematic approach of formulating closed-form expressions for the approximate eigenvalue sensitivities of a linear structure carrying lumped attachments. Specifically, a set of general purpose formulas is developed that readily yields the effects of various parameters on the eigencharacteristics of the system directly. Because the assumed-modes method was used to formulate the frequency equation of a combined system consisting of an arbitrarily supported linear structure carrying any number of lumped attachments, the proposed scheme of calculating the eigenvalues of the combined system offers numerous advantages. First, Eqs. (5) and (14) are simple to code. Given the eigenfunctions of the host linear structure, the parameters for the lumped attachments and their locations, Eqs. (5) and (14) can be easily programmed and solved either graphically or numerically using any existing root solvers. Second, the proposed approach can be used to determine the eigenvalue sensitivities with respect to any physical parameter of the lumped attachments. Third, while the ith eigenvalue sensitivity depends on all of the eigenvalues and eigenfunctions of the host structure, which are known explicitly, it only depends on the ith eigenvalue of the combined system and not on its eigenvectors nor the other eigenvalues. Fourth, the formulation of the present study is general, and the results can be easily extended to encompass any one- or two-dimensional linear structure onto which one or more lumped elements are attached. Finally, the eigenvalue sensitivities can be used in conjunction with the Taylor series expansion to compute the perturbed eigenvalues of a slightly modified system without performing a potentially costly reanalysis.

To fix ideas without any loss of generality, let the linear structure consist of a uniform Euler–Bernoulli beam. The eigenfunctions used in the assumed-modes method depend on the boundary conditions of the bare beam. For a uniform simply supported beam, its normalized (with respect to ρ) eigenfunctions are given by [22]

$$\phi_i(x) = \sqrt{\frac{2}{\rho L}}\sin\frac{i\pi x}{L} \tag{22}$$

such that

$$M_i = 1, \qquad K_i = (i\pi)^4 EI/(\rho L^4) \tag{23}$$

For a uniform fixed–free beam, an expression for its normalized eigenfunctions that remains well-conditioned for the first 200 modes is given by [23]

$$\phi_i(x) = \frac{1}{\sqrt{\rho L}}[a_i(x) - b_i(x) + (1+\nu_i)c_i(x) - \nu_i d_i(x)] \tag{24}$$

where

$$a_i(x) = e^{-\beta_i x}, \qquad b_i(x) = \cos\beta_i x, \qquad c_i(x) = \sin\beta_i x$$
$$d_i(x) = \sinh\beta_i x, \qquad \nu_i = \frac{-(e^{-\beta_i L} + \cos\beta_i L + \sin\beta_i L)}{\cosh\beta_i L + \cos\beta_i L} \tag{25}$$

and $\beta_i L$ satisfies the following transcendental equation,

$$\cos\beta_i L \cosh\beta_i L = -1 \tag{26}$$

such that the generalized masses and stiffnesses of the beam are

$$M_i = 1, \qquad K_i = (\beta_i L)^4 EI/(\rho L^4) \tag{27}$$

In the following sections, Eqs. (5) and (14) will be used to determine the natural frequencies or eigenvalues of various combined systems, and the proposed method will be used to compute the corresponding sensitivities. Whenever possible, the results will be compared with those obtained exactly. In all of the subsequent numerical examples, N will be set to be 20 to ensure consistency and to guarantee sufficient accuracy. Finally, to assess the quality of the approximate results, the percent error is given, defined as

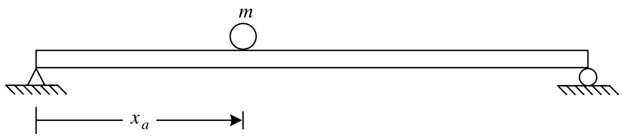

Fig. 1 Simply supported beam with lumped mass $m = 0.5\rho L$ at $x_a = 0.3L$.

$$\%\ \text{error} = |\text{exact} - \text{approximate}|/|\text{exact}| \times 100\%$$

A. Simply Supported Beam with Lumped Mass

Consider the system shown in Fig. 1, which consists of a simply supported beam with a lumped mass m at x_a. For this combined system, the exact frequency equation can be readily obtained (see Appendix C), and one can compute the exact sensitivities of the natural frequencies with respect to the lumped mass parameter and its attachment location by direct partial differentiations. While conceptually straightforward, formulating the characteristic determinant and expanding it to obtain the frequency equation is nontrivial. Using Table 1 and setting $\lambda = j\omega$, the approximate frequency equation for the system of Fig. 1 is given by

$$1 - m\omega^2 \sum_{i=1}^{N} \frac{\phi_i^2(x_a)}{\omega_{hi}^2 - \omega^2} = 0 \tag{28}$$

whose solution readily yields the natural frequencies of the combined system. Using Table 2, the natural frequency sensitivities with respect to m are

$$\frac{\partial\omega}{\partial m} = -\frac{\partial g/\partial m}{\partial g/\partial\omega} = -\left(\omega \sum_{i=1}^{N} \frac{\phi_i^2(x_a)}{\omega_{hi}^2 - \omega^2}\right) \Bigg/ \left(2m \sum_{i=1}^{N} \frac{\omega_{hi}^2 \phi_i^2(x_a)}{(\omega_{hi}^2 - \omega^2)^2}\right) \tag{29}$$

Using the proposed method, a complete set of the natural frequencies is not needed to obtain each natural frequency sensitivity. Note also that the natural frequency sensitivities are inversely proportional to the lumped mass m. Thus, as m increases, the natural frequencies will become less sensitive to slight variations in m. Using Table 2, one can also readily obtain the natural frequency sensitivities with respect to x_a:

$$\frac{\partial\omega}{\partial x_a} = -\frac{\partial g/\partial x_a}{\partial g/\partial\omega} = -\left(\omega \sum_{i=1}^{N} \frac{\phi_i(x_a)\phi_i'(x_a)}{\omega_{hi}^2 - \omega^2}\right) \Bigg/ \left(\sum_{i=1}^{N} \frac{\omega_{hi}^2 \phi_i^2(x_a)}{(\omega_{hi}^2 - \omega^2)^2}\right) \tag{30}$$

Tables 3 and 4 show the approximate and exact natural frequencies and their sensitivities for the system of Fig. 1, with respect to both m and x_a, for $m = 0.5\rho L$ and $x_a = 0.3L$. Note that the approximate natural frequencies and sensitivities agree very well with the exact solution. In addition, note that the natural frequency sensitivities with respect to the mass parameter are all negative, implying that an increase in the lumped mass value will cause a decrease in natural frequencies, the result of which is completely consistent with physical intuition. The sensitivities of the natural frequencies with respect to the attachment location, however, can be positive or negative. Thus, for the chosen set of system parameters, some natural frequencies will increase while others will decrease if the attachment location is slightly varied. It is important to emphasize that one can easily formulate expressions for the approximate natural frequency sensitivities using the results of Table 2, whereas the exact sensitivities requires one to first solve for the exact frequency equation and then perform the desired partial differentiations, which is a daunting task even for this simple system.

Table 3 First five nondimensionalized natural frequencies of a simply supported beam with a lumped mass $m = 0.5\rho L$ at $x_a = 0.3L$[a]

	Equation (5)	Exact	% error
ω_1	7.6140×10^0	7.6139×10^0	2.54×10^{-4}
ω_2	3.1798×10^1	3.1798×10^1	2.19×10^{-3}
ω_3	8.7142×10^1	8.7140×10^1	1.22×10^{-3}
ω_4	1.4613×10^2	1.4611×10^2	1.46×10^{-2}
ω_5	2.1381×10^2	2.1373×10^2	3.56×10^{-2}

[a]The natural frequencies are nondimensionalized by dividing by $\sqrt{EI/(\rho L^4)}$.

Table 4 First five nondimensionalized natural frequency sensitivities for the system of Table 3

	Proposed method	Exact	% error
$\frac{\partial\omega_1}{\partial m} \Big/ \sqrt{\frac{EI}{\rho^3 L^6}}$	-3.1519×10^0	-3.1519×10^0	1.22×10^{-3}
$\frac{\partial\omega_2}{\partial m} \Big/ \sqrt{\frac{EI}{\rho^3 L^6}}$	-6.5019×10^0	-6.5028×10^0	1.34×10^{-2}
$\frac{\partial\omega_3}{\partial m} \Big/ \sqrt{\frac{EI}{\rho^3 L^6}}$	-1.3238×10^0	-1.3254×10^0	1.23×10^{-1}
$\frac{\partial\omega_4}{\partial m} \Big/ \sqrt{\frac{EI}{\rho^3 L^6}}$	-9.4021×10^0	-9.4310×10^0	3.06×10^{-1}
$\frac{\partial\omega_5}{\partial m} \Big/ \sqrt{\frac{EI}{\rho^3 L^6}}$	-1.5653×10^1	-1.5703×10^1	3.23×10^{-1}
$\frac{\partial\omega_1}{\partial x_a} \Big/ \sqrt{\frac{EI}{\rho L^6}}$	-6.6686×10^0	-6.6688×10^0	1.92×10^{-3}
$\frac{\partial\omega_2}{\partial x_a} \Big/ \sqrt{\frac{EI}{\rho L^6}}$	3.4204×10^1	3.4201×10^1	8.66×10^{-3}
$\frac{\partial\omega_3}{\partial x_a} \Big/ \sqrt{\frac{EI}{\rho L^6}}$	8.9820×10^1	8.9862×10^1	4.66×10^{-2}
$\frac{\partial\omega_4}{\partial x_a} \Big/ \sqrt{\frac{EI}{\rho L^6}}$	-3.7036×10^2	-3.7098×10^2	1.66×10^{-1}
$\frac{\partial\omega_5}{\partial x_a} \Big/ \sqrt{\frac{EI}{\rho L^6}}$	1.9956×10^2	1.9979×10^2	1.15×10^{-1}

B. Simply Supported Beam with Viscoelastic Solid

Figure 2 shows a simply supported beam with the Maxwell model of a viscoelastic solid, which consists of a linear spring and damper in series. For this combined system, the exact frequency equation can be obtained after some lengthy algebra (see Appendix D), and the exact eigenvalue sensitivities can be computed by direct differentiations. Using Table 1, the approximate frequency equation for the combined system is

$$1 + \frac{ck\lambda}{c\lambda + k} \sum_{i=1}^{N} \frac{\phi_i^2(x_a)}{\omega_{hi}^2 + \lambda^2} = 0 \tag{31}$$

and the approximate eigenvalue sensitivities with respect to the system parameters and attachment location are formulated using the results shown in Appendix B and noting that

$$\frac{\partial\lambda}{\partial k} = -\frac{\partial g/\partial k}{\partial g/\partial\lambda}, \qquad \frac{\partial\lambda}{\partial c} = -\frac{\partial g/\partial c}{\partial g/\partial\lambda}, \qquad \frac{\partial\lambda}{\partial x_a} = -\frac{\partial g/\partial x_a}{\partial g/\partial\lambda} \tag{32}$$

For definiteness, the system parameters are $k = 20EI/L^3$, $c = 5\sqrt{EI\rho/L^2}$, and $x_a = 0.31L$. Tables 5 and 6 shows the eigenvalues and their sensitivities for the system of Fig. 2; for brevity, only the first three eigenvalues and their sensitivities are presented. Note the excellent agreement between the approximate eigenvalues and their sensitivities and those calculated exactly.

Table 5 First three nondimensionalized eigenvalues of a simply supported beam with the Maxwell model for a viscoelastic solid[a]

	Equation (5)	Exact	% error
λ_1	$-3.8835\times10^{-1}+1.1023\times10^{1}j$	$-3.8834\times10^{-1}+1.1023\times10^{1}j$	8.42×10^{-5}
λ_2	$-4.4380\times10^{-2}+3.9917\times10^{1}j$	$-4.4379\times10^{-2}+3.9917\times10^{1}j$	9.45×10^{-6}
λ_3	$-4.8803\times10^{-4}+8.8837\times10^{1}j$	$-4.8803\times10^{-4}+8.8837\times10^{1}j$	1.04×10^{-7}

[a]The system parameters are $k=20EI/L^3$, $c=5\sqrt{EI\rho/L^2}$, and $x_a=0.31L$. The natural frequencies are nondimensionalized by dividing by $\sqrt{EI/(\rho L^4)}$.

Table 6 First three nondimensionalized eigenvalue sensitivities for the system of Table 5

	PROPOSED METHOD	Exact	% error
$\frac{\partial\lambda_1}{\partial k}\Big/\sqrt{\frac{L^2}{\rho EI}}$	$-3.1770\times10^{-2}+4.4378\times10^{-2}j$	$-3.1769\times10^{-2}+4.4378\times10^{-2}j$	1.56×10^{-3}
$\frac{\partial\lambda_2}{\partial k}\Big/\sqrt{\frac{L^2}{\rho EI}}$	$-4.4149\times10^{-3}+2.1704\times10^{-2}j$	$-4.4147\times10^{-3}+2.1704\times10^{-2}j$	1.71×10^{-3}
$\frac{\partial\lambda_3}{\partial k}\Big/\sqrt{\frac{L^2}{\rho EI}}$	$-4.9032\times10^{-5}+5.3972\times10^{-4}j$	$-4.9031\times10^{-5}+5.3971\times10^{-4}j$	1.71×10^{-3}
$\frac{\partial\lambda_1}{\partial c}\Big/\frac{1}{\rho L}$	$6.5959\times10^{-2}+4.3791\times10^{-2}j$	$6.5958\times10^{-2}+4.3790\times10^{-2}j$	1.48×10^{-3}
$\frac{\partial\lambda_2}{\partial c}\Big/\frac{1}{\rho L}$	$8.7018\times10^{-3}+1.7600\times10^{-3}j$	$8.7017\times10^{-3}+1.7599\times10^{-3}j$	1.70×10^{-3}
$\frac{\partial\lambda_3}{\partial c}\Big/\frac{1}{\rho L}$	$9.7207\times10^{-5}+8.8304\times10^{-6}j$	$9.7205\times10^{-5}+8.8302\times10^{-6}j$	1.71×10^{-3}
$\frac{\partial\lambda_1}{\partial x_a}\Big/\sqrt{\frac{EI}{\rho L^6}}$	$-1.4562\times10^{0}+4.9337\times10^{0}j$	$-1.4562\times10^{0}+4.9337\times10^{0}j$	3.37×10^{-4}
$\frac{\partial\lambda_2}{\partial x_a}\Big/\sqrt{\frac{EI}{\rho L^6}}$	$2.0817\times10^{-1}-2.1305\times10^{0}j$	$2.0816\times10^{-1}-2.1305\times10^{0}j$	1.90×10^{-3}
$\frac{\partial\lambda_3}{\partial x_a}\Big/\sqrt{\frac{EI}{\rho L^6}}$	$4.1180\times10^{-2}-9.0816\times10^{-1}j$	$4.1179\times10^{-2}-9.0815\times10^{-1}j$	9.18×10^{-4}

C. Cantilever Beam with Tip Mass and In-Span Simple Support

Consider the system analyzed in [16], which consists of a cantilever beam with a tip mass of m and an in-span simple support at x_a, as shown in Fig. 3. The frequency equation of the system can be formulated using Eq. (17) and letting $\lambda=j\omega$, $\sigma_1=k$, $x_1=x_a$, $\sigma_2=-m\omega^2$, and $x_2=L$. After some algebra, one obtains the following frequency equation:

$$0=1-km\omega^2\left(\sum_{i=1}^{N}\frac{\phi_i^2(x_a)}{\omega_{hi}^2-\omega^2}\right)\left(\sum_{i=1}^{N}\frac{\phi_i^2(L)}{\omega_{hi}^2-\omega^2}\right)+k\left(\sum_{i=1}^{N}\frac{\phi_i^2(x_a)}{\omega_{hi}^2-\omega^2}\right)-m\omega^2\left(\sum_{i=1}^{N}\frac{\phi_i^2(L)}{\omega_{hi}^2-\omega^2}\right)+km\omega^2\left(\sum_{i=1}^{N}\frac{\phi_i(x_a)\phi_i(L)}{\omega_{hi}^2-\omega^2}\right)^2 \tag{33}$$

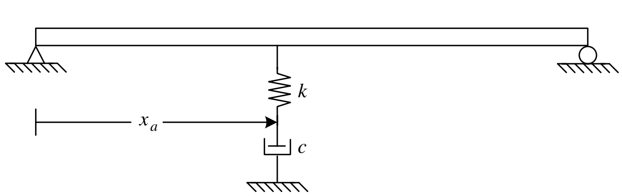

Fig. 2 Simply supported beam with the Maxwell model for a viscoelastic solid (a linear spring and damper in series). The system parameters are $k=20EI/L^3$, $c=5\sqrt{EI\rho/L^2}$, and $x_a=0.31L$.

Dividing Eq. (33) by $-km\omega^2$ and letting $k\to\infty$ to model the simple support, one finds the frequency equation for the system shown in Fig. 2 to be

$$0=\left(\sum_{i=1}^{N}\frac{\phi_i^2(x_a)}{\omega_{hi}^2-\omega^2}\right)\left[\left(\sum_{i=1}^{N}\frac{\phi_i^2(L)}{\omega_{hi}^2-\omega^2}\right)-\frac{1}{m\omega^2}\right]-\left(\sum_{i=1}^{N}\frac{\phi_i(x_a)\phi_i(L)}{\omega_{hi}^2-\omega^2}\right)^2 \tag{34}$$

which is identical to Eq. (28) derived in [16]. To compute the sensitivities of the natural frequency using the implicit function theorem, a function g is defined that is given by the right-hand-side of Eq. (34). Differentiating g with respect to x_a, m, and ω, the sensitivities of the natural frequencies with respect to the location of the in-span support and to the tip mass are given by

$$\frac{\partial\omega}{\partial x_a}=-\frac{\partial g/\partial x_a}{\partial g/\partial\omega} \tag{35}$$

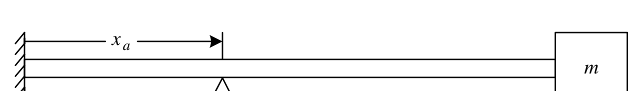

Fig. 3 Cantilever beam with tip mass of $m=0.5\rho L$ and an in-span support at $x_a=0.1L$.

$$\frac{\partial \omega}{\partial m} = -\frac{\partial g/\partial m}{\partial g/\partial \omega} \tag{36}$$

$$\frac{\partial g}{\partial x_a} = -2a_1a_2 + 2a_3a_4 - \frac{2}{m\omega^2}a_4 \tag{37}$$

$$\frac{\partial g}{\partial m} = \frac{1}{m^2\omega^2}a_6 \tag{38}$$

$$\frac{\partial g}{\partial \omega} = 2\omega a_5a_6 + 2\omega a_3a_7 - 4\omega a_1a_8 + \frac{2}{m\omega^3}a_6 - \frac{2}{m\omega}a_7 \tag{39}$$

where a_i are defined as follows:

$$a_1 = \sum_{i=1}^{N}\frac{\phi_i(x_a)\phi_i(L)}{\omega_{hi}^2-\omega^2}, \qquad a_2 = \sum_{i=1}^{N}\frac{\phi_i'(x_a)\phi_i(L)}{\omega_{hi}^2-\omega^2}$$
$$a_3 = \sum_{i=1}^{N}\frac{\phi_i^2(L)}{\omega_{hi}^2-\omega^2}, \qquad a_4 = \sum_{i=1}^{N}\frac{\phi_i(x_a)\phi_i'(x_a)}{\omega_{hi}^2-\omega^2}$$
$$a_5 = \sum_{i=1}^{N}\frac{\phi_i^2(L)}{(\omega_{hi}^2-\omega^2)^2}, \qquad a_6 = \sum_{i=1}^{N}\frac{\phi_i^2(x_a)}{\omega_{hi}^2-\omega^2}$$
$$a_7 = \sum_{i=1}^{N}\frac{\phi_i^2(x_a)}{(\omega_{hi}^2-\omega^2)^2}, \qquad a_8 = \sum_{i=1}^{N}\frac{\phi_i(x_a)\phi_i(L)}{(\omega_{hi}^2-\omega^2)^2} \tag{40}$$

Equation (35) yields the sensitivity of the natural frequencies with respect to the location of the in-span support. It can be shown to be identical to Eq. (37) in [16], which gives the sensitivity of ω^2 with respect to x_a, by noting that

$$2\omega\frac{\partial \omega}{\partial x_a} = \frac{\partial}{\partial x_a}(\omega^2) \tag{41}$$

Tables 7 and 8 show the first five natural frequencies and their sensitivities with respect to x_a and m for the system of Fig. 3, for $m = 0.5\rho L$ and $x_a = 0.1L$. For comparison, the exact natural frequencies and their sensitivities are also evaluated by solving the exact frequency equation [see Eq. (8) of [16]] and by partial differentiating the exact frequency equation with respect to x_a and m. Note the excellent agreement between the approximate and exact natural frequencies and sensitivities.

Table 7 First five nondimensionalized natural frequencies of a cantilever beam with a tip mass $m = 0.5\rho L$ and an in-span simple support at $x_a = 0.1L$[a]

	Equation (14)	Exact	% error
ω_1	2.2979×10^0	2.2976×10^0	1.53×10^{-2}
ω_2	1.9762×10^1	1.9757×10^1	2.35×10^{-2}
ω_3	6.0866×10^1	6.0842×10^1	4.03×10^{-2}
ω_4	1.2541×10^2	1.2534×10^2	5.10×10^{-2}
ω_5	2.1374×10^2	2.1356×10^2	8.46×10^{-2}

[a]The natural frequencies are nondimensionalized by dividing by $\sqrt{EI/(\rho L^4)}$.

Table 8 First five nondimensionalized natural frequency sensitivities for the system of Table 7

	Proposed method	Exact	% error
$\frac{\partial\omega_1}{\partial x_a}\Big/\sqrt{\frac{EI}{\rho L^6}}$	3.1371×10^0	3.1459×10^0	2.81×10^{-1}
$\frac{\partial\omega_2}{\partial x_a}\Big/\sqrt{\frac{EI}{\rho L^6}}$	3.3138×10^1	3.3216×10^1	2.37×10^{-1}
$\frac{\partial\omega_3}{\partial x_a}\Big/\sqrt{\frac{EI}{\rho L^6}}$	1.0802×10^2	1.0845×10^2	3.98×10^{-1}
$\frac{\partial\omega_4}{\partial x_a}\Big/\sqrt{\frac{EI}{\rho L^6}}$	2.3142×10^2	2.3175×10^2	1.40×10^{-1}
$\frac{\partial\omega_5}{\partial x_a}\Big/\sqrt{\frac{EI}{\rho L^6}}$	4.0237×10^2	4.0455×10^2	5.38×10^{-1}
$\frac{\partial\omega_1}{\partial m}\Big/\sqrt{\frac{EI}{\rho^3 L^6}}$	-1.5924×10^0	-1.5921×10^0	1.80×10^{-2}
$\frac{\partial\omega_2}{\partial m}\Big/\sqrt{\frac{EI}{\rho^3 L^6}}$	-2.6111×10^0	-2.6108×10^0	1.34×10^{-2}
$\frac{\partial\omega_3}{\partial m}\Big/\sqrt{\frac{EI}{\rho^3 L^6}}$	-3.3410×10^0	-3.3425×10^0	4.66×10^{-2}
$\frac{\partial\omega_4}{\partial m}\Big/\sqrt{\frac{EI}{\rho^3 L^6}}$	-3.6280×10^0	-3.6334×10^0	1.50×10^{-1}
$\frac{\partial\omega_5}{\partial m}\Big/\sqrt{\frac{EI}{\rho^3 L^6}}$	-3.7867×10^0	-3.8047×10^0	4.73×10^{-1}

D. Simply Supported Beam with Multiple Attachments

Consider a more complicated system shown in Fig. 4, which consists of a simply supported beam with a grounded translational spring $k = 5EI/L^3$ at $x_1 = 0.2L$, a damped oscillator with a rigid-body degree of freedom with system parameters $m_1 = 0.12\rho L$, $c_1 = 1\sqrt{EI\rho/L^2}$, and $k_1 = 4EI/L^3$ at $x_2 = 0.6L$ and a grounded torsional spring $k_t = 10EI/L$ at $x_3 = 0.75L$. Table 9 shows the eigenvalues of the nominal system (or the unperturbed eigenvalues), calculated using Eq. (14) and the finite element method (FEM), where the beam is discretized into 100 uniform elements. Note the excellent agreement between the two approaches. This damped system will be referred to as the nominal system. For brevity, Table 10 shows the approximate sensitivities of the fundamental eigenvalue with respect to all the physical parameters obtained using the proposed method. Because the exact frequency equation for this system is difficult to formulate, the exact eigenvalue sensitivities are not available. Fortunately, the accuracy of the approximate eigenvalue sensitivities can be inferred by using them to calculate approximations to the perturbed eigenvalues when slight modifications are introduced and subsequently comparing these approximate perturbed eigenvalues with those obtained using the finite element method [24]. To validate the eigenvalue sensitivities of Table 10, the following perturbations are introduced:

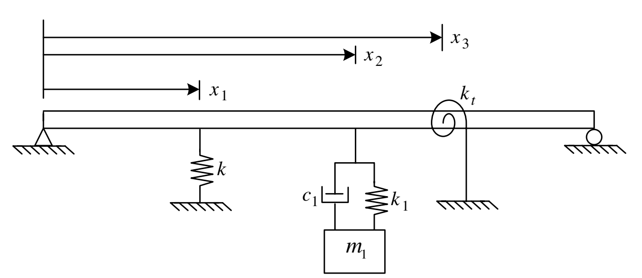

Fig. 4 Simply supported beam with a grounded translational spring at $x_1 = 0.2L$, a damped oscillator with a rigid-body degree of freedom at $x_2 = 0.6L$, and a grounded torsional spring at $x_3 = 0.75L$. The system parameters are $k = 5EI/L^3$, $m_1 = 0.12\rho L$, $c_1 = 1\sqrt{EI\rho/L^2}$, $k_1 = 4EI/L^3$, and $k_t = 10EI/L$.

Table 9 First five nondimensionalized nominal eigenvalues of a simply supported beam with a grounded translational spring at $x_1 = 0.2L$, a damped oscillator with a rigid-body degree of freedom at $x_2 = 0.6L$, and a grounded torsional spring at $x_3 = 0.75L$[a]

	Equation (14)	FEM ($N_e = 100$)	% error
λ_{10}	$-4.2116 \times 10^0 + 4.1765 \times 10^0 j$	$-4.2128 \times 10^0 + 4.1794 \times 10^0 j$	5.21×10^{-2}
λ_{20}	$-8.1376 \times 10^{-1} + 1.2746 \times 10^1 j$	$-8.1473 \times 10^{-1} + 1.2652 \times 10^1 j$	7.43×10^{-1}
λ_{30}	$-3.3614 \times 10^{-1} + 3.9540 \times 10^1 j$	$-3.3615 \times 10^{-1} + 3.9540 \times 10^1 j$	1.60×10^{-5}
λ_{40}	$-2.9362 \times 10^{-1} + 9.2496 \times 10^1 j$	$-2.9602 \times 10^{-1} + 9.2374 \times 10^1 j$	1.33×10^{-1}
λ_{50}	$-1.0244 \times 10^0 + 1.6635 \times 10^2 j$	$-1.0223 \times 10^0 + 1.6602 \times 10^2 j$	1.98×10^{-1}

[a]The system parameters are $k = 5EI/L^3$, $m_1 = 0.12\rho L$, $c_1 = 1\sqrt{EI\rho/L^2}$, $k_1 = 4EI/L^3$, and $k_t = 10EI/L$. The eigenvalues are nondimensionalized by dividing by $\sqrt{EI/(\rho L^4)}$.

Table 10 Nondimensionalized sensitivities of the fundamental eigenvalue for the system of Table 9

	Proposed method
$\dfrac{\partial\lambda_1}{\partial k}\Big/\sqrt{\dfrac{L^2}{\rho EI}}$	$2.8183 \times 10^{-4} - 5.2923 \times 10^{-4} j$
$\dfrac{\partial\lambda_1}{\partial m_1}\Big/\sqrt{\dfrac{EI}{\rho^3 L^6}}$	$3.8551 \times 10^1 + 9.8723 \times 10^{-1} j$
$\dfrac{\partial\lambda_1}{\partial c_1}\Big/\dfrac{1}{\rho L}$	$-4.9714 \times 10^0 - 4.3806 \times 10^0 j$
$\dfrac{\partial\lambda_1}{\partial k_1}\Big/\sqrt{\dfrac{L^2}{\rho EI}}$	$7.5103 \times 10^{-2} + 1.1146 \times 10^0 j$
$\dfrac{\partial\lambda_1}{\partial k_t}\Big/\sqrt{\dfrac{L^2}{\rho EI}}$	$2.5293 \times 10^{-3} - 5.8600 \times 10^{-3} j$
$\dfrac{\partial\lambda_1}{\partial x_1}\Big/\sqrt{\dfrac{EI}{\rho L^6}}$	$1.2452 \times 10^{-2} - 2.4204 \times 10^{-2} j$
$\dfrac{\partial\lambda_1}{\partial x_2}\Big/\sqrt{\dfrac{EI}{\rho L^6}}$	$8.4241 \times 10^{-2} - 2.6727 \times 10^{-1} j$
$\dfrac{\partial\lambda_1}{\partial x_3}\Big/\sqrt{\dfrac{EI}{\rho L^6}}$	$2.8052 \times 10^{-1} - 6.6212 \times 10^{-1} j$

$$\Delta k = 0.75EI/L^3, \qquad \Delta m_1 = -0.01\rho L$$
$$\Delta c_1 = -0.1\sqrt{EI\rho/L^2}, \qquad \Delta k_1 = 0.5EI/L^3$$
$$\Delta k_t = -2EI/L, \qquad \Delta x_1 = 0.02L$$
$$\Delta x_2 = -0.01L, \qquad \Delta x_3 = 0.01L$$

and the approximations of the perturbed eigenvalues are obtained using a first-order approximation and the sensitivity expressions [see Eq. (21)]. Table 11 shows the first five approximate perturbed eigenvalues obtained using the proposed method. For comparison, the finite element method (the beam is discretized into 100 uniform elements) is also used to compute the eigenvalues of the modified system. Note how well the results track one another, thus indirectly validating the accuracy of the eigenvalue sensitivities. Using the proposed scheme to calculate approximations of the perturbed eigenvalues of the modified system only requires the eigenvalues and the eigenvalue sensitivities of the nominal system. The finite element method, however, requires one to resolve a generalized eigenvalue problem of size 402×402. Thus, the proposed scheme can be efficiently used to analyze the effects of small perturbations on the eigenvalues of the system.

E. Simply Supported Rectangular Plate with Lumped Mass

Because the assumed-modes method is used to derive the equations of motion, the proposed scheme is general and can be applied to other linear structures as long as the eigensolutions of the bare structures are available. Consider the two-dimensional system of Fig. 5, which consists of a simply supported rectangular plate with sides a and b carrying m located at (x_a, y_a). In this case, the normalized eigenfunctions (with respect to ρ) of the simply supported plate are given by [25]

$$\phi_{pq}(x, y) = \frac{2}{\sqrt{ab\rho}} \sin\left(\frac{p\pi x}{a}\right) \sin\left(\frac{q\pi y}{b}\right) \tag{42}$$

such that

$$M_{pq} = 1, \qquad K_{pq} = \frac{D}{\rho}\left(\frac{p^2}{a^2} + \frac{q^2}{b^2}\right)^2 \pi^4 \tag{43}$$

where $D = Eh^3/(12(1-\nu^2))$. For this two-dimensional problem, the results of Table 2 still remain valid, although slight modifications need to be made to account for the fact that the problem now depends on two spatial coordinates. For the system of Fig. 5, the sensitivities of the natural frequencies are

$$\frac{\partial\omega}{\partial m} = -\left(\omega \sum_{p=1}^{r}\sum_{q=1}^{s} \frac{\phi_{pq}^2(x_a, y_a)}{\omega_{hpq}^2 - \omega^2}\right) \Bigg/ \left(2m \sum_{p=1}^{r}\sum_{q=1}^{s} \frac{\omega_{hpq}^2 \phi_{pq}^2(x_a, y_a)}{(\omega_{hpq}^2 - \omega^2)^2}\right) \tag{44}$$

where $\omega_{hpq}^2 = K_{pq}/M_{pq}$. Note that Eq. (44) is nearly identical to Eq. (29), except that the single summation becomes a double summation, and the eigenfunctions of the bare linear structure are now evaluated at (x_a, y_a). The natural frequency sensitivities with respect to x_a and y_a are similarly obtained. For definiteness, assume

Table 11 First five nondimensionalized eigenvalues for the perturbed system[a]

	Equation (21)	FEM ($N_e = 100$)	% error
λ_1	$-4.0651 \times 10^0 + 5.1689 \times 10^0 j$	$-4.0536 \times 10^0 + 5.1441 \times 10^0 j$	4.17×10^{-1}
λ_2	$-8.1774 \times 10^{-1} + 1.2651 \times 10^1 j$	$-8.2532 \times 10^{-1} + 1.2517 \times 10^1 j$	1.08×10^0
λ_3	$-2.4931 \times 10^{-1} + 3.9589 \times 10^1 j$	$-2.5568 \times 10^{-1} + 3.9609 \times 10^1 j$	5.43×10^{-2}
λ_4	$-3.7836 \times 10^{-1} + 9.1284 \times 10^1 j$	$-3.7047 \times 10^{-1} + 9.1289 \times 10^1 j$	1.05×10^{-2}
λ_5	$-8.8294 \times 10^0 + 1.6477 \times 10^2 j$	$-8.5835 \times 10^0 + 1.6444 \times 10^2 j$	1.98×10^{-1}

[a]The nominal system parameters are identical to those of Table 9, and the modifications are $\Delta k = 0.75EI/L^3$, $\Delta m_1 = -0.01\rho L$, $\Delta c_1 = -0.1\sqrt{EI\rho/L^2}$, $\Delta k_1 = 0.5EI/L^3$, $\Delta k_t = -2EI/L$, $\Delta x_1 = 0.02L$, $\Delta x_2 = -0.01L$, and $\Delta x_3 = 0.01L$. The eigenvalues are nondimensionalized by dividing by $\sqrt{EI/(\rho L^4)}$.

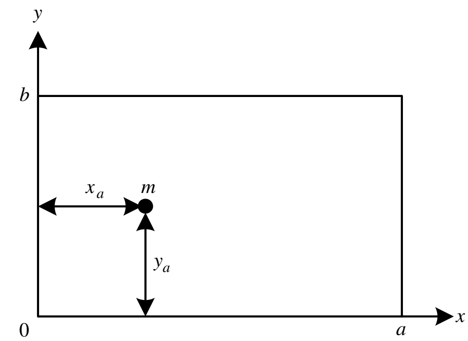

Fig. 5 Simply supported rectangular plate with lengths a and b carrying a lumped mass m at the point (x_a, y_a). The system parameters are $a/b = 2/3$, $m = 0.2ab\rho$, $x_a/a = 0.2$, and $y_a/b = 0.7$.

that the system parameters are $a/b = 2/3, m = 0.2ab\rho, x_a/a = 0.2$, and $y_a/b = 0.7$. To validate the sensitivities obtained using the proposed method, the exact sensitivities are obtained by taking the partial derivatives of the exact frequency equation of a simply supported plate carrying a lumped mass [see Eq. (12) from [26]]. For the proposed method, $r = s = 20$, and the exact solution is approximated by using $r = s = 100$. Table 12 shows the first three natural frequencies and Table 13 shows the first three natural frequency sensitivities with respect to the mass parameter and its attachment locations. Note how well the proposed method tracks the exact solution.

F. Discussion on Limitations of the Approach

A few words on the limitations of the proposed method to determine eigenvalue sensitivities are warranted. The results of Table 2 and Appendix B are only applicable when the eigenvalues of the combined system are distinct from those of the host structure. When one or more eigenvalues of the combined system and the bare structure coincide, the implicit function theorem can still be used to determine the eigenvalue sensitivities. In this case, however, the function g must be slightly modified. For the single-attachment case, g is redefined as [see Eq. (2)]

$$g = \left(\prod_{i=1}^{N}(\lambda^2 M_i + K_i)\right)\left(1 + \sigma \sum_{i=1}^{N} \frac{u_i^2(x_a)}{\lambda^2 M_i + K_i}\right) \tag{45}$$

and for the multiple-attachments case, g becomes

$$g = \left(\prod_{i=1}^{N}(\lambda^2 M_i + K_i)\right) \det[B] \tag{46}$$

where the elements of $[B]$ are given in Eq. (15). Incidentally, for a linear structure carrying a translational (or rotational) lumped

Table 12 First three nondimensionalized natural frequencies of a simply supported rectangular plate with sides a and b carrying a lumped mass m at point (x_a, y_a)[a]

	Frequency equation ($r = s = 20$)	Exact ($r = s = 100$)	% error
ω_1	1.9287×10^1	1.9284×10^1	1.45×10^{-2}
ω_2	3.5540×10^1	3.5519×10^1	5.94×10^{-2}
ω_3	5.3285×10^1	5.3230×10^1	1.03×10^{-1}

[a]The system parameters are $a/b = 2/3$, $m = 0.2ab\rho$, $x_a/a = 0.2$, and $y_a/b = 0.7$. The natural frequencies are nondimensionalized by dividing by $\sqrt{\rho a^2 b^2/D}$.

Table 13 First three nondimensionalized natural frequency sensitivities for the system of Table 12

	Proposed method ($r = s = 20$)	Exact ($r = s = 100$)	% error
$\frac{\partial\omega_1}{\partial m}\Big/\sqrt{\frac{\rho^3 a^4 b^4}{D}}$	-1.0795×10^1	-1.0820×10^1	2.26×10^{-1}
$\frac{\partial\omega_2}{\partial m}\Big/\sqrt{\frac{\rho^3 a^4 b^4}{D}}$	-2.3755×10^1	-2.3868×10^1	4.77×10^{-1}
$\frac{\partial\omega_3}{\partial m}\Big/\sqrt{\frac{\rho^3 a^4 b^4}{D}}$	-2.7487×10^1	-2.7533×10^1	4.89×10^{-1}
$\frac{\partial\omega_1}{\partial x_a}\Big/\sqrt{\frac{\rho a^4 b^2}{D}}$	-1.6493×10^1	-1.6513×10^1	1.19×10^{-1}
$\frac{\partial\omega_2}{\partial x_a}\Big/\sqrt{\frac{\rho a^4 b^2}{D}}$	-2.3207×10^1	-2.3204×10^1	1.12×10^{-2}
$\frac{\partial\omega_3}{\partial x_a}\Big/\sqrt{\frac{\rho a^4 b^2}{D}}$	1.4692×10^1	1.4841×10^1	9.99×10^{-1}
$\frac{\partial\omega_1}{\partial y_a}\Big/\sqrt{\frac{\rho a^2 b^4}{D}}$	7.1511×10^0	7.1651×10^0	1.96×10^{-1}
$\frac{\partial\omega_2}{\partial y_a}\Big/\sqrt{\frac{\rho a^2 b^4}{D}}$	-2.0210×10^1	-2.0172×10^1	1.89×10^{-1}
$\frac{\partial\omega_3}{\partial y_a}\Big/\sqrt{\frac{\rho a^2 b^4}{D}}$	1.5108×10^1	1.5292×10^1	1.20×10^0

element whose attachment location coincides with a point of zero displacement (or slope) for the ith mode of the host structure, one can easily show that the sensitivities of the eigenvalue that corresponds to $j\omega_{hi}$ with respect to x_a and the parameters of the lumped attachment are identically zero (see Appendix E). Finally, the proposed method is only applicable when the eigenvalues are distinct. For systems with repeated eigenvalues, the readers are encouraged to refer to the methods developed in [27–30].

IV. Conclusions

An efficient formulation is proposed that immediately yields the approximate sensitivities of the eigenvalues of an arbitrarily supported linear structure carrying any number of lumped attachments, including point masses, rotary inertias, grounded translational or torsional springs, grounded translational or torsional viscous dampers, undamped and damped oscillators with no rigid-body degree of freedom, undamped and damped oscillators with a rigid-body degree of freedom, or the Maxwell model for a viscoelastic solid. The proposed scheme exploits the implicit function theorem and leads to the following advantages: it accommodates a linear structure with any boundary conditions; it can be modified to compute the eigenvalue sensitivities of a linear structure carrying any combination of miscellaneous undamped and damped attachments; and finally, it can be used to determine the perturbed eigenvalues of a combined system subjected to slight modifications by using the Taylor series expansion. Numerical experiments validate the utility of the proposed method, and excellent agreement was found between the approximate and exact eigenvalue sensitivities.

Appendix A: Implicit Function Theorem

Consider a function f of two variables x and y. Let $f(x, y)$ and $(\partial f/\partial y)(x, y)$ be continuous in a region D of the (x, y) plane, and let (x_0, y_0) be a point inside D, where $f(x_0, y_0) = 0$ and $(\partial f/\partial y)(x_0, y_0) \neq 0$. Then the implicit function theorem [21] states the following:

1) There is a region R within D containing (x_0, y_0) such that for any (x, y) in R there is a unique y that satisfies $f(x, y) = 0$.

2) If $y = g(x)$, then $y_0 = g(x_0)$, with $f(x, g(x)) = 0$, and $g(x)$ is continuous inside R.

3) Finally, if $(\partial f/\partial x)(x, y)$ is continuous in D, then $g(x)$ is differentiable in R and

$$g'(x_0) = -\left(\frac{\partial f}{\partial x}(x_0, y_0)\right) \bigg/ \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)$$

The implicit function theorem gives conditions that ensure the existence of a continuously differentiable inverse function. The theorem has a more general form that involves functions $f(x_1, x_2, \ldots, x_n)$ of n variables, though this will not be presented here. The interested reader can find the generalizations of the implicit function theorem in [21].

Appendix B: Expressions of g and Partial Derivatives for Oscillator Attachments or Viscoelastic Solid

1) For a translational damped oscillator with no rigid degree of freedom, g and its partial derivatives are listed below. For a rotational damped oscillator, replace $(k, c, m, \phi_i(x))$ by $(k_r, c_r, I, \phi_i'(x))$:

$$g = 1 + (k + c\lambda + m\lambda^2) \sum_{i=1}^{N} \frac{\phi_i^2(x_a)}{\lambda^2 + \omega_{hi}^2}$$

$$\frac{\partial g}{\partial \lambda} = \sum_{i=1}^{N} \frac{[\omega_{hi}^2(c + 2m\lambda) - \lambda(c\lambda + 2k)]\phi_i^2(x_a)}{(\lambda^2 + \omega_{hi}^2)^2}$$

$$\frac{\partial g}{\partial k} = \sum_{i=1}^{N} \frac{\phi_i^2(x_a)}{\lambda^2 + \omega_{hi}^2} \qquad \frac{\partial g}{\partial c} = \lambda \sum_{i=1}^{N} \frac{\phi_i^2(x_a)}{\lambda^2 + \omega_{hi}^2}$$

$$\frac{\partial g}{\partial m} = \lambda^2 \sum_{i=1}^{N} \frac{\phi_i^2(x_a)}{\lambda^2 + \omega_{hi}^2}$$

$$\frac{\partial g}{\partial x_a} = 2(k + c\lambda + m\lambda^2) \sum_{i=1}^{N} \frac{\phi_i(x_a)\phi_i'(x_a)}{\lambda^2 + \omega_{hi}^2}$$

2) For a translational damped oscillator with a rigid degree of freedom, g and its partial derivatives are listed below. For a rotational damped oscillator, replace $(k, c, m, \phi_i(x))$ by $(k_r, c_r, I, \phi_i'(x))$:

$$g = k + c\lambda + m\lambda^2 + (k + c\lambda)m\lambda^2 \sum_{i=1}^{N} \frac{\phi_i^2(x_a)}{\lambda^2 + \omega_{hi}^2}$$

$$\frac{\partial g}{\partial \lambda} = c + 2m\lambda + \sum_{i=1}^{N} \frac{[(2km\lambda + 3cm\lambda^2)\omega_{hi}^2 + cm\lambda^4]\phi_i^2(x_a)}{(\lambda^2 + \omega_{hi}^2)^2}$$

$$\frac{\partial g}{\partial k} = 1 + m\lambda^2 \sum_{i=1}^{N} \frac{\phi_i^2(x_a)}{\lambda^2 + \omega_{hi}^2} \qquad \frac{\partial g}{\partial c} = \lambda + m\lambda^3 \sum_{i=1}^{N} \frac{\phi_i^2(x_a)}{\lambda^2 + \omega_{hi}^2}$$

$$\frac{\partial g}{\partial m} = \lambda^2 + \lambda^2(k + c\lambda) \sum_{i=1}^{N} \frac{\phi_i^2(x_a)}{\lambda^2 + \omega_{hi}^2}$$

$$\frac{\partial g}{\partial x_a} = 2m\lambda^2(k + c\lambda) \sum_{i=1}^{N} \frac{\phi_i(x_a)\phi_i'(x_a)}{\lambda^2 + \omega_{hi}^2}$$

3) For the Maxwell model of a viscoelastic solid, g and its partial derivatives are listed below:

$$g = c\lambda + k + ck\lambda \sum_{i=1}^{N} \frac{\phi_i^2(x_a)}{\omega_{hi}^2 + \lambda^2}$$

$$\frac{\partial g}{\partial \lambda} = c + ck \sum_{i=1}^{N} \frac{(\omega_{hi}^2 - \lambda^2)\phi_i^2(x_a)}{(\omega_{hi}^2 + \lambda^2)^2} \qquad \frac{\partial g}{\partial k} = 1 + c\lambda \sum_{i=1}^{N} \frac{\phi_i^2(x_a)}{\omega_{hi}^2 + \lambda^2}$$

$$\frac{\partial g}{\partial c} = \lambda + k\lambda \sum_{i=1}^{N} \frac{\phi_i^2(x_a)}{\omega_{hi}^2 + \lambda^2} \qquad \frac{\partial g}{\partial x_a} = 2ck\lambda \sum_{i=1}^{N} \frac{\phi_i(x_a)\phi_i'(x_a)}{\omega_{hi}^2 + \lambda^2}$$

Appendix C: Exact Frequency Equation for a Simply Supported Beam with Lumped Mass

The exact frequency equation of the system shown in Fig. 1 is

$$f(\beta L) = \det[D] = 0$$

The matrix $[D]$ is defined as

$$[D] = \begin{bmatrix} 0 & 0 & \cos(\beta L) & \sin(\beta L) & \cosh(\beta L) & \sinh(\beta L) \\ 0 & 0 & -\cos(\beta L) & -\sin(\beta L) & \cosh(\beta L) & \sinh(\beta L) \\ -\sin(\beta x_a) & -\sinh(\beta x_a) & \cos(\beta x_a) & \sin(\beta x_a) & \cosh(\beta x_a) & \sinh(\beta x_a) \\ -\cos(\beta x_a) & -\cosh(\beta x_a) & -\sin(\beta x_a) & \cos(\beta x_a) & \sinh(\beta x_a) & \cosh(\beta x_a) \\ \sin(\beta x_a) & -\sinh(\beta x_a) & -\cos(\beta x_a) & -\sin(\beta x_a) & \cosh(\beta x_a) & \sinh(\beta x_a) \\ \alpha_1 & \alpha_2 & \sin(\beta L x_a) & -\cos(\beta x_a) & \sinh(\beta x_a) & \cosh(\beta x_a) \end{bmatrix}$$

where

$$\alpha_1 = -\hat{m}\beta L \sin(\beta x_a) + \cos(\beta x_a),$$

$$\alpha_2 = -\hat{m}\beta L \sinh(\beta x_a) - \cosh(\beta x_a)$$

$\hat{m} = m/(\rho L)$, and the ith natural frequency for the system of Fig. 1 is given by $\omega_i = (\beta_i L)^2 \sqrt{EI/(\rho L^4)}$.

Appendix D: Exact Frequency Equation for a Simply Supported Beam with Viscoelastic Solid

The exact frequency equation of the system shown in Fig. 2 is

$$f(\beta L) = \det[D] = 0$$

The matrix $[D]$ is defined as

$$[D] = \begin{bmatrix} \sin(\beta x_a) & \sinh(\beta x_a) & -\cos(\beta x_a) & -\sin(\beta x_a) & -\cosh(\beta x_a) & -\sinh(\beta x_a) \\ \cos(\beta x_a) & \cosh(\beta x_a) & \sin(\beta x_a) & -\cos(\beta x_a) & -\sinh(\beta x_a) & -\cosh(\beta x_a) \\ -\sin(\beta x_a) & \sinh(\beta x_a) & \cos(\beta x_a) & \sin(\beta x_a) & -\cosh(\beta x_a) & -\sinh(\beta x_a) \\ \alpha_1 & \alpha_2 & \beta L \sin(\beta x_a) & -\beta L \cos(\beta x_a) & \beta L \sinh(\beta x_a) & \beta L \cosh(\beta x_a) \\ 0 & 0 & \cos(\beta L) & \sin(\beta L) & \cosh(\beta L) & \sinh(\beta L) \\ 0 & 0 & -\cos(\beta L) & -\sin(\beta L) & \cosh(\beta L) & \sinh(\beta L) \end{bmatrix}$$

where

$$\alpha_1 = \beta L \cos(\beta x_a) + \hat{k} \sin(\beta x_a) \frac{\hat{c} j}{\hat{k} + \hat{c}(\beta L)^2 j}$$

$$\alpha_2 = -\beta L \cosh(\beta x_a) + \hat{k} \sinh(\beta x_a) \frac{\hat{c} j}{\hat{k} + \hat{c}(\beta L)^2 j}$$

$\hat{k} = k/(EI/L^3)$, $\hat{c} = c/\sqrt{EI\rho/L^2}$, and the ith eigenvalue for the system of Fig. 2 is given by $\lambda_i = (\beta_i L)^2 j \sqrt{EI/(\rho L^4)}$.

Appendix E: Eigenvalue Sensitivities When Attachment Location Coincides with Eigenfunction Node

Consider a linear structure carrying either a lumped translational or rotational attachment. Assume that the attachment location x_a coincides with a point of zero displacement or zero slope of the nth eigenfunction of the bare structure, in which case $u_n(x_a) = 0$ [for a translational element, $u_n(x_a) = \phi_n(x_a) = 0$; for a rotational element, $u_n(x_a) = \phi_n'(x_a) = 0$]. If the eigenfunctions of the host structure are normalized with respect to its mass per unit length, $M_i = 1$ and $K_i = \omega_{hi}^2$, and the frequency equation is given by Eq. (4). Expanding Eq. (4), one obtains

$$\prod_{i=1}^{N} (\lambda^2 + \omega_{hi}^2) + \sigma \sum_{i=1}^{N} \left[u_i^2(x_a) \prod_{\substack{k=1\\k\neq i}}^{N} (\lambda^2 + \omega_{hk}^2) \right] = 0$$

When $u_n(x_a) = 0$, then $\lambda^2 + \omega_{hn}^2 = 0$, and this term can be factored out of the left-hand side of the above equation. Thus, $\lambda = j\omega_{hn}$ is an eigenvalue of the combined system. To calculate the sensitivities of this particular eigenvalue, the implicit function theorem can be used. Defining the left-hand side of the above equation to be g, the eigenvalue sensitivities are given by

$$\frac{\partial \lambda}{\partial \sigma} = -\frac{\partial g}{\partial \sigma} \Big/ \frac{\partial g}{\partial \lambda} \qquad \frac{\partial \lambda}{\partial x_a} = -\frac{\partial g}{\partial x} \Big/ \frac{\partial g}{\partial \lambda}$$

where

$$\frac{\partial g}{\partial \sigma} = \sum_{i=1}^{N} \left[u_i^2(x_a) \prod_{\substack{k=1\\k\neq i}}^{N} (\lambda^2 + \omega_{hk}^2) \right]$$

$$\frac{\partial g}{\partial x_a} = 2\sigma \sum_{i=1}^{N} \left[u_i(x_a) u_i'(x_a) \prod_{\substack{k=1\\k\neq i}}^{N} (\lambda^2 + \omega_{hk}^2) \right]$$

$$\frac{\partial g}{\partial \lambda} = \sum_{i=1}^{N} \left[2\lambda \prod_{\substack{k=1\\k\neq i}}^{N} (\lambda^2 + \omega_{hk}^2) \right] + \frac{\partial \sigma}{\partial \lambda} \sum_{i=1}^{N} \left[u_i^2(x_a) \prod_{\substack{k=1\\k\neq i}}^{N} (\lambda^2 + \omega_{hk}^2) \right] + \sigma \sum_{i=1}^{N} \left[u_i^2(x_a) \sum_{\substack{k=1\\k\neq i}}^{N} \left(2\lambda \prod_{\substack{l=1\\l\neq k}}^{N} (\lambda^2 + \omega_{hl}^2) \right) \right]$$

Because $u_n(x_a) = 0$ and $\lambda^2 + \omega_{hn}^2 = 0$, one finds that $\partial g/\partial \alpha_s = (\partial g/\partial \sigma)(\partial \sigma/\partial \alpha_s) = 0$, $\partial g/\partial x_a = 0$, and $\partial g/\partial \lambda \neq 0$ for $\lambda = j\omega_{hn}$. Thus, the sensitivities of this particular eigenvalue with respect to α_s and x_a are zero.

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